Solution Sheet 8

Exercise 8.1

Let $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\frac{\partial u}{\partial t}(t, y) = b\left(t, u(t, y)\right), \qquad \frac{\partial u}{\partial y}(t, y) = \sigma\left(t, u(t, y)\right)$$

where b, σ are also smooth (in space and time). Show that the process X defined by

$$X_t := u(t, W_t)$$

solves the SDE

$$dX_t = \left(b(t, X_t) + \frac{1}{2}\sigma(t, X_t)\frac{\partial \sigma}{\partial x}(t, X_t)\right) dt + \sigma(t, X_t)dW_t$$

for $X_0 = u(0, 0)$.

Proof. We wish to apply the Itô Formula for $u(t, W_t)$. Using the given derivative relations, then

$$\frac{\partial u}{\partial y}(t, W_t) = \sigma\left(t, u(t, W_t)\right)$$

which is an adapted and continuous process hence progressively measurable, whilst also satisfying pathwise integrability in time. Additionally,

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}(t, y) &= \frac{\partial}{\partial y}\left(\sigma(t, u(t, y))\right)\\ &= \frac{\partial \sigma}{\partial y}\left(t, u(t, y)\right)\frac{\partial u}{\partial y}(t, y)\\ &= \frac{\partial \sigma}{\partial y}\left(t, u(t, y)\right)\sigma\left(t, u(t, y)\right).\end{aligned}$$

Thus we can apply the Itô Formula to achieve that

$$\begin{aligned}u(t, W_t) = u(0, W_0) &+ \int_0^t \sigma(s, u(s, W_s))dW_s + \int_0^t b(s, u(s, W_s))ds\\ &+ \frac{1}{2}\int_0^t \frac{\partial \sigma}{\partial y}(s, u(s, W_s))\sigma(s, u(s, W_s))d[W]_s\end{aligned}$$

for which using that $[W]_s = s$ and substituting in X gives the result.

□

Exercise 8.2

Solve the SDE

$$dX_t = \left(\sqrt{1 + X_t^2} + \frac{1}{2}X_t\right) dt + \sqrt{1 + X_t^2}dW_t$$

for $X_0 = x$ deterministic. *Hint: Ansatz available upon request!*

Proof. The idea is to use Exercise 8.1 with the function $u(t, y) = \sinh(c+t+y)$ making the Ansatz $X_t = \sinh(C + t + W_t)$ where $\sinh(x) = \frac{e^x - e^{-x}}{2}$. We can arrive at this Ansatz by comparing to Exercise 8.1 and seeing that we would like $\sigma(t, x) = b(t, x) = \sqrt{1 + x^2}$, and using the hyperbolic trigonometric identities. Indeed we use the function

$$u(t, y) = \sinh(C + t + y)$$

so that

$$\frac{\partial u}{\partial t}(t, y) = \cosh(C + t + y) = \sqrt{1 + (\sinh(C + t + y))^2} = \sqrt{1 + (u(t, y))^2}$$

and similarly

$$\frac{\partial u}{\partial y}(t, y) = \sqrt{1 + (u(t, y))^2}$$

which does indeed give us, in the notation of Exercise 8.1, that $b(t, x) = \sigma(t, x) = \sqrt{1 + x^2}$ so

$$\frac{\partial \sigma}{\partial x}(t, x) = \frac{x}{\sqrt{1 + x^2}}.$$

Thus applying Exercise 8.1 with $X_t = \sinh(C + t + W_t)$ gives the result, choosing C such that $x = u(0, 0) = \sinh(C)$ so $C = \sinh^{-1}(x)$.

□

Exercise 8.3

Similarly to Exercise 1.3, we introduce the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

where b and σ are assumed Lipschitz and of linear growth such that the existence and uniqueness of the SDE holds, as well as the flow map Φ where $\Phi_{s,t}(u)$ is the solution of the SDE at time t with 'initial condition' $X_s = u$, and for $s = 0$ we simply use $\Phi_t(u)$. Prove the following:

1. For every $s \geq 0$, the processes $\Phi_{s,\cdot}(u)$ and $\Phi_{\cdot - s}(u)$, starting from s, have the same distribution.

2. For every $s \geq 0$, the processes $\Phi_{s,\cdot}(\Phi_s(u))$ and $\Phi_\cdot(u)$, starting from s, are indistinguishable.

3. For every $t \geq 0$, there exists a constant C_t such that for all u, v,

$$\mathbb{E}\left[\sup_{s\in[0,t]} |\Phi_s(u) - \Phi_s(v)|^2\right] \leq C_t \mathbb{E}\left[|u - v|^2\right]$$

Proof.

1. We write out the identities satisfied by the processes, that is

$$\Phi_{s,t}(u) = u + \int_s^t b(\Phi_{s,r}(u))dr + \int_s^t \sigma(\Phi_{s,r})dW_r$$

and

$$\Phi_{t-s}(u) = u + \int_0^{t-s} b(\Phi_r(u))dr + \int_0^{t-s} \sigma(\Phi_r)dW_r.$$

Using the substitution $l = r - s$, then

$$\Phi_{s,t}(u) = u + \int_0^{t-s} b(\Phi_{s,s+l}(u))dl + \int_0^{t-s} \sigma(\Phi_{s,s+l}(u))dW_{s+l}.$$

Now we introduce the new Brownian Motion

$$\tilde{W}_l = W_{s+l} - W_s$$

(one can check direction that this is a Brownian Motion for the filtration $(\tilde{\mathcal{F}}_l) = (\mathcal{F}_{s+l})$) such that

$$\int_0^{t-s} \sigma(\Phi_{s,s+l}(u))dW_{s+l} = \int_0^{t-s} \sigma(\Phi_{s,s+l}(u))d\left(\tilde{W}_l + W_s\right) = \int_0^{t-s} \sigma(\Phi_{s,s+l}(u))d\tilde{W}_l.$$

In particular $\Phi_{s,\cdot}(u)$ and $\Phi_{\cdot-s}(u)$ satisfy the same SDE just with a different Brownian Motion, so by the uniqueness and hence uniqueness in law of solutions so SDEs, the two processes have the same distribution.

2. We observe that

$$\Phi_{s,t}(\Phi_s(u)) = \Phi_s(u) + \int_s^t b(\Phi_{s,r}(\Phi_s(u)))dr + \int_s^t \sigma(\Phi_{s,r}(\Phi_s(u)))dW_r$$

and

$$\Phi_t(u) = \Phi_s(u) + \int_s^t b(\Phi_r(u))dr + \int_s^t \sigma(\Phi_r(u))dW_r$$

hence

$$\Phi_{s,t}(\Phi_s(u)) - \Phi_t(u) = \int_s^t b(\Phi_{s,r}(\Phi_s(u))) - b(\Phi_r(u))dr + \int_s^t \sigma(\Phi_{s,r}(\Phi_s(u))) - \sigma(\Phi_r(u))dW_r.$$

We can now take the square of both sides, using that $(a+b)^2 \leq 2a^2 + 2b^2$, and Hölder's Inequality in the time integral to see that

$$|\Phi_{s,t}(\Phi_s(u)) - \Phi_t(u)|^2 \leq 2(t-s)\int_s^t |b(\Phi_{s,r}(\Phi_s(u))) - b(\Phi_r(u))|^2 dr + \left|\int_s^t \sigma(\Phi_{s,r}(\Phi_s(u))) - \sigma(\Phi_r(u))dW_r\right|^2.$$

Now we may take expectation and apply the Itô Isometry, followed by using the Lipschitz properties of b and σ, to deduce the existence of a constant C dependent only on $t-s$ and the Lipschitz coefficients of b and σ such that

$$\mathbb{E}\left[|\Phi_{s,t}(\Phi_s(u)) - \Phi_t(u)|^2\right] \leq C\int_s^t \mathbb{E}\left[|\Phi_{s,r}(\Phi_s(u)) - \Phi_r(u)|^2\right] dr$$

having also used Fubini-Tonelli to interchange expectation and integration. Now, exactly as we did in Exercise 1.3, we apply the Grönwall Inequality with $\alpha = 0$ as the above reads

$$\psi_t \leq C\int_s^t \psi_r dr$$

with $\psi_t = \mathbb{E}\left[|\Phi_{s,t}(\Phi_s(u)) - \Phi_t(u)|^2\right]$, to deduce that

$$\mathbb{E}\left[|\Phi_{s,t}(\Phi_s(u)) - \Phi_t(u)|^2\right] = 0.$$

This implies that

$$|\Phi_{s,t}(\Phi_s(u)) - \Phi_t(u)|^2 = 0$$

$\mathbb{P} - a.s.$, and in particular there is a set Ω_t of full probability on which the identity holds. To show indistinguishability we need a set of full probability such that the identity holds for all t. In particular we may take all rational t_n and define

$$\tilde{\Omega} = \bigcap_{s \le t_n \in \mathbb{Q}} \Omega_{t_n}$$

which is a set of full probability such that $|\Phi_{s,t}(\Phi_s(u)) - \Phi_t(u)|^2 = 0$ holds for all rational times everywhere on $\tilde{\Omega}$. By continuity of the processes then this equality must hold for all times, hence the result.

3. Similarly to the above, we have that

$$\begin{aligned}|\Phi_s(u) - \Phi_s(v)|^2 \le 3|u-v|^2 + 3s\int_0^s |b(\Phi_r(u)) - b(\Phi_r(v))|^2 dr \\ + 3\left|\int_0^s \sigma(\Phi_r(u)) - \sigma(\Phi_r(v)) dW_r\right|^2.\end{aligned}$$

Now we take supremum over $s \in [0,t]$ on both sides to see that

$$\begin{aligned}\sup_{s\in[0,t]} |\Phi_s(u) - \Phi_s(v)|^2 \le 3|u-v|^2 + 3t\int_0^t |b(\Phi_r(u)) - b(\Phi_r(v))|^2 dr \\ + 3\sup_{s\in[0,t]}\left|\int_0^s \sigma(\Phi_r(u)) - \sigma(\Phi_r(v)) dW_r\right|^2.\end{aligned}$$

followed by expectation, then applying Doob's Maximal Inequality and the Itô Isometry, as well as the Lipschitz assumptions, then

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} |\Phi_s(u) - \Phi_s(v)|^2\right] &\le 3\mathbb{E}\left[|u-v|^2\right] + C\int_0^t \mathbb{E}|\Phi_r(u) - \Phi_r(v)|^2 dr \\ &\le 3\mathbb{E}\left[|u-v|^2\right] + C\int_0^t \mathbb{E}\left[\sup_{s\in[0,r]} |\Phi_s(u) - \Phi_s(v)|^2\right] dr\end{aligned}$$

after which applying Grönwall's Inequality gives the result.

□

Exercise 8.4

Consider the Ornstein-Uhlenbeck SDE

$$dX_t = -aX_t dt + \sigma dW_t$$

for constants a, σ, and posed for an initial condition $X_0 \sim N(x_0, v_0)$ independent of W.

1. Show that X satisfies, $\mathbb{P} - a.s.$,

$$X_t = e^{-at}X_0 + \sigma e^{-at}\int_0^t e^{as}dW_s.$$

2. Show that

$$\mathrm{Cov}(X_s, X_t) = e^{-a(t+s)}\left[v_0 + \frac{\sigma^2}{2a}(e^{2a(t\wedge s)} - 1)\right].$$

Proof.

1. We apply the Itô Formula for $f(t,x) = e^{at}x$, following the same Ansatz as for Exercise 1.5. Then

$$\frac{\partial}{\partial t}f(t,x) = ae^{at}x, \qquad \frac{\partial}{\partial x}f(t,x) = e^{at}, \qquad \frac{\partial^2}{\partial x^2}f(t,x) = 0$$

so by the Itô Formula for $f(t, X_t)$,

$$\begin{aligned} e^{at}X_t &= e^{a\cdot 0}X_0 + \int_0^t ae^{as}X_s ds + \int_0^t e^{as}dX_s \\ &= X_0 + \int_0^t ae^{as}X_s ds + \int_0^t e^{as}\left(-aX_s + \sigma dW_s\right) \\ &= X_0 + \sigma\int_0^t e^{as}dW_s \end{aligned}$$

for which multiplying by e^{-at} gives the result.

2. Of course $\mathrm{Cov}(X_s, X_t) = \mathbb{E}(X_sX_t) - \mathbb{E}(X_s)\mathbb{E}(X_t)$ and from the first part, using that the stochastic integral is a martingale hence of constant (zero) expectation,

$$\mathbb{E}(X_t) = e^{-at}x_0$$

therefore

$$\mathbb{E}(X_s)\mathbb{E}(X_t) = e^{-a(t+s)}x_0^2. \tag{1}$$

Additionally,

$$\begin{aligned} \mathbb{E}(X_sX_t) &= e^{-a(t+s)}\mathbb{E}\left[X_0^2 + \sigma^2\int_0^t e^{ar}dW_r\int_0^s e^{ar}dW_r + X_0\sigma\int_0^t e^{ar}dW_r + X_0\sigma\int_0^s e^{ar}dW_r\right] \\ &= e^{-a(t+s)}\mathbb{E}\left[X_0^2 + \sigma^2\int_0^t e^{ar}dW_r\int_0^s e^{ar}dW_r\right] \end{aligned}$$

due to independence of X_0 and W, hence independence of X_0 and the stochastic integral. We use that for a general square integrable martingale M with $s < t$,

$$\mathbb{E}(M_tM_s) = \mathbb{E}\left[\mathbb{E}(M_tM_s|\mathcal{F}_s\right] = \mathbb{E}\left[M_s\mathbb{E}(M_t|\mathcal{F}_s\right] = \mathbb{E}(M_s^2).$$

So applying this result to the stochastic integral, followed by the Itô Isometry,

$$\mathbb{E}\left[\int_0^t e^{ar}dW_r\int_0^s e^{ar}dW_r\right] = \int_0^s e^{2ar}dr = \frac{1}{2a}(e^{2as} - 1).$$

Thus,

$$\mathbb{E}(X_s X_t) = e^{-a(t+s)} \left[\mathbb{E}(X_0^2) + \frac{\sigma^2}{2a}(e^{2as} - 1) \right]$$

from which we subtract (1) and observe that

$$v_0 = \mathbb{E}(X_0^2) - x_0^2$$

to conclude.

□

Exercise 8.5

Let M be a continuous local martingale. Define the process N by $N_t = e^{M_t - \frac{1}{2}[M]_t}$, known as the exponential martingale.

1. Prove that N is a continuous local martingale.

2. Show that N is a genuine martingale if and only if $\mathbb{E}(N_t) = 1$ for all $t \geq 0$.

Proof.

1. Define the semi-martingale Y by $Y_t = M_t - \frac{1}{2}[M]_t$ and function $f(x) = e^x$, so applying the Itô Formula for $f(Y)$,

$$\begin{aligned} e^{Y_t} &= e^{Y_0} + \int_0^t e^{Y_s} dY_s + \frac{1}{2} \int_0^t e^{Y_s} d[Y]_s \\ &= e^{M_0} + \int_0^t e^{Y_s} dM_s - \frac{1}{2} \int_0^t e^{Y_s} d[M]_s + \frac{1}{2} \int_0^t e^{Y_s} d[M]_s \\ &= e^{M_0} + \int_0^t e^{Y_s} dM_s \end{aligned}$$

   where we have used that $[Y] = [M]$. This is a continuous local martingale.

2. If N is a martingale it has constant expectation, so $\mathbb{E}(N_t) = 1$. We prove the converse. As N is a non-negative continuous local martingale it is a super-martingale, as for (τ_n) a localising sequence of stopping times and by Fatou,

$$\mathbb{E}(N_t | \mathcal{F}_s) \leq \lim_{n\to\infty} \mathbb{E}\left(N_t^{\tau_n} | \mathcal{F}_s\right) = \lim_{n\to\infty} N_s^{\tau_n} = N_s.$$

   We now use that a super-martingale of constant expectation is a martingale to conclude.

□